Lect #1

1. Effective, conformal & superconformal QFTs
2. Moduli spaces, parameter spaces, symmetries, dualities
3. Coulomb branch geometries of 4d $\mathcal{N}=2$ susy QFTs
4. Vertex algebras of 4d $\mathcal{N}=2$ SCFTs

Goal: Illustrate how physics (QFTs) informs math questions
- basic arguments physicists use to reason about QFTs (lect 1 & 2)
- Highlight a few open questions, esp. those w. precise math. formulations (lect 3 & 4)

M.	10-11 am	ILC	S331
Tu.	11-12 noon	ILC	S131
Th.	11-12 noon	ILC	S331
F.	10-11 am	ILC	S131

1. Effective, conformal, & superconformal QFTs
   A. Def'n QFT
   B. Renormalization group & effective FTs
   C. Conformal FT
   D. Supersymmetry & superconformal FTs

A. "QFT" ~ "Wightman axioms"

(0) • QM: $\mathcal{H}$ Hilbert space $\simeq \mathbb{C}^{n\to\infty}$ w/ pos. def. inner prod $\langle a|b\rangle \in \mathbb{C}$

observables = self-adj ops $M \in \mathrm{End}(\mathcal{H})$, symms $\in U(\mathcal{H})$ isometries

(1) • "space-time symmetry" = Poincaré (cf. QM)

Lie alg $iso(d-1,1)$

$$\begin{cases} P_\mu \circ P_\nu = 0 & \mu,\nu \in \{1 \dots d\} \quad \text{transl} \\ M_{\mu\nu} \circ M_{\rho\sigma} = \dots so(d-1,1) & \text{Lorentz} \\ M_{\mu\nu} \circ P_\rho = (\mathfrak{M}_{\mu\nu})_\rho{}^\sigma P_\sigma & \end{cases}$$

[illegible]

d-dim'l Lor. irrep

ad

generate Lie-group

$$\mathcal{U}(a,\omega) \doteq \exp\{x^\mu P_\mu + \omega^{\mu\nu} M_{\mu\nu}\}$$

$x \in \mathbb{R}^{d-1,1} \doteq$ Mink. st., $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

diag $= \{+ \dots + -\}$

(data) • local operators (fields) $\mathcal{O} \in \mathrm{End}(\mathcal{H})$

$$\begin{cases} \mathcal{O}(x) \doteq e^{x^\mu P_\mu} \circ \mathcal{O} & (\mathcal{O} = \mathcal{O}(0)) \\ M_{\mu\nu} \circ \mathcal{O}_i = (R_{\mu\nu})_i{}^j \mathcal{O}_j & \end{cases}$$

Ad

Can't'l comm.

finite-dim'l Lorentz rep.

(2) • $\mathcal{O}_1(x_1) \circ \mathcal{O}_2(x_2) = 0$ if $x_{12}^2 > 0$ "causality" "micro-locality"

(3) • $\exists\ |v\rangle \in \mathcal{H}$ s.t. $P|v\rangle = M|v\rangle = 0$ "vacua"

& $\mathrm{spec}(P_0) \geq 0$ "stability"

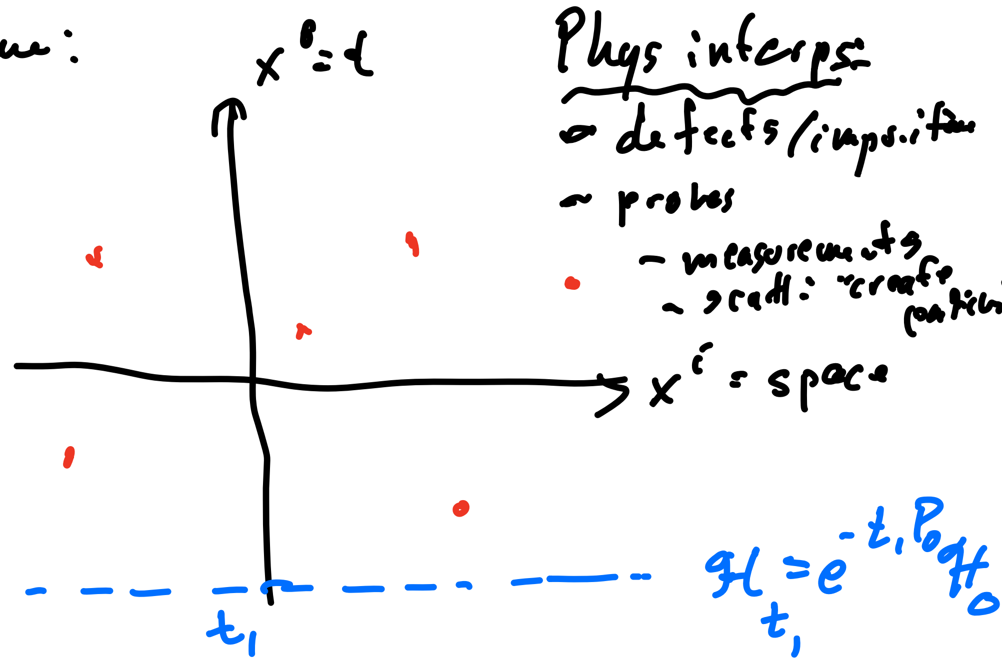
[(4) • Enough $\mathcal{O}$'s so that $\mathrm{Poly}(\mathcal{O}\text{'s})|v\rangle$ dense in $\mathcal{H}$.]

(5) • $\exists$ stress-energy tensor $\leftrightarrow$ "local FT"

$\exists$ hermitian loc. op $\begin{cases} T_{\mu\nu}(x) = T_{\nu\mu}(x) \\ \partial^\mu T_{\mu\nu}(x) = 0 \end{cases}$

s.t. $\oint_{\Sigma_{d-1}}$ generate Poincaré transfs on operators in interior of Σ_{d-1}.

Space-time picture:



Phys interps:
- defects/impurities
- probes
  - measurements
  - scatt: "create particles"

(Defn) • time-ordered correlators:

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_v \doteq \langle v| T[\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)]|v\rangle \in \mathbb{C}$$

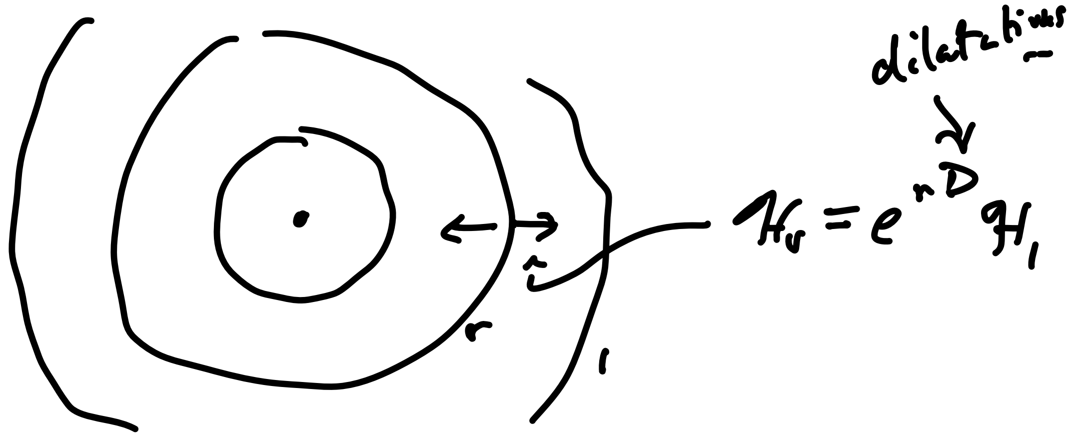
("absolute" FT)

→ reconstruct* $\mathcal{H}$ & $\mathcal{O}_i$ etc ops "quantization"

→ E.g. Path integral & cutting-sewing axioms

* w/ bc's on correlators: cluster decomp

• Different quantization foliations (see esp. euclidean) give different $\mathcal{H}'$ & $\mathcal{O}_i'$ e.g. "radial"



— "QFT" generalize from "unitary, Poincaré, local":

- continue to euclidean space
- non-unitary: $\langle\psi|\chi\rangle$ on $\mathcal{H}$ not pos. def.
- $\langle\mathcal{O}_1 \dots \mathcal{O}_n\rangle \in \mathbb{C}^n$ $n>1$: "relative" FT
- put on other fixed s-t geometries / change of spaces
- quasi-local: $\nexists\, T_{\mu\nu}(x)$, or even any $\mathcal{O}_i(x)$
- turn on gravity! (can be done: nature, string theory)

— Even within unitary Poincaré local QFT

- known that not fully specified by local correlators
- add (at least) "extended operators"

$\mathcal{O}(\Sigma^n)$ n-dim'l submanfld of s-t.

- but axiomatization of their correlators not known ...
- e.g. "gauge theories"

- Unitary Poincaré local QFT "ill-defined" bc
  a) can't prove known constructions satisfy axioms
  b) don't have gen. construction / axioms

  - continuum limit of lattice QM
  - semiclassical ($\mathcal{L}$, PI ...) resummation

B. Local QFTs as RG flows

- $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \doteq G(x_1 \ldots x_n) \quad (x_i \neq x_j)$

$$" G(\lambda x_1 \ldots \lambda x_n) \overset{\lambda \gg 1}{\sim} \lambda^{-\Delta} \tilde{G}_0(x_1 \ldots x_n) + \lambda^{-\Delta'} e^{-m\lambda} \tilde{G}(x) + \ldots "$$

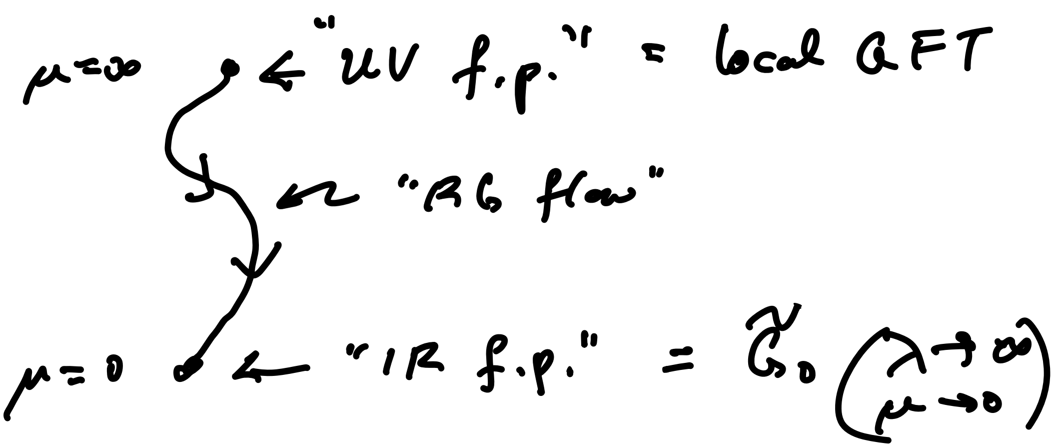
$$\Delta > 0$$

- Idea: $\tilde{G}_0$ "simpler" than G bc have "scaled away" microscopic details: $\tilde{G}_0$ only probes leading behavior at distances $\gg \lambda$.
  Furthermore, if only ask questions (corr's) for separations $\gg \lambda$, then $\tilde{G}_0$ have all properties of (local unitary Poinc) QFT.

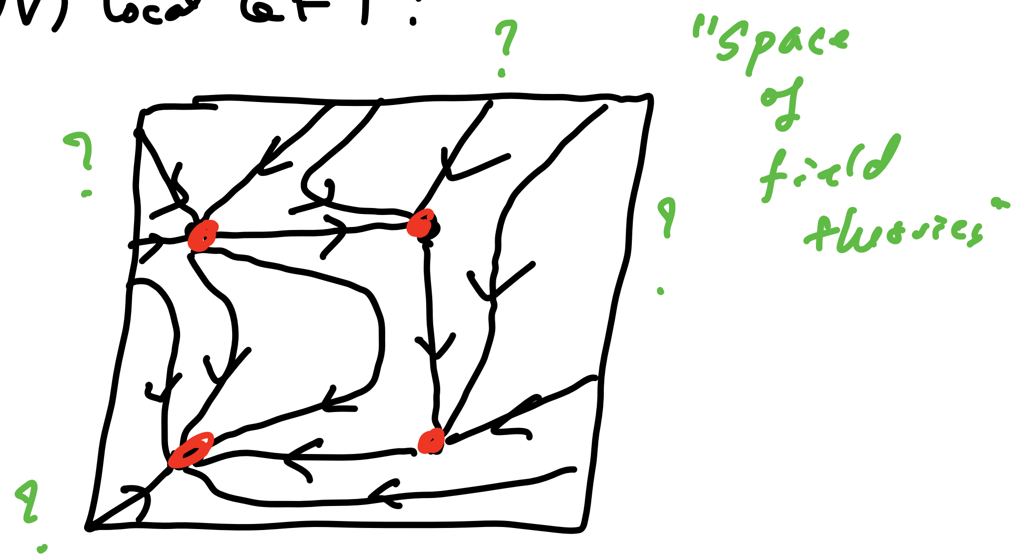
- $\tilde{G}_0 \longleftrightarrow$ Wilsonian effective FT @ scale $\mu \doteq \frac{1}{\lambda}$

- Really $\tilde{G}_0 \leftrightarrow$ "IR" eff FT, i.e. at scale $\mu = \frac{1}{\lambda} \to 0$.

- Picture: "space of eff. FTs"



- But if limit $\lambda \to \infty$ $\mu \to 0$ gives nontrivial $\tilde{G}(x)$'s $\Rightarrow$ can re-interpret as (UV) local QFT:



Lect #2 7/18/23

1. C. CFTs

- F.p. theories by def'n are scale-invariant. Have extra symmetry: Poincaré + dilatations

$$D \circ P_\mu = P_\mu$$
$$D \circ M_{\mu\nu} = 0$$
$$D \circ \mathcal{O}_i = \Delta_i \mathcal{O}_i \qquad \text{(diagonalize ?)}$$

(scaling) dim'n > 0

- If $T_{\mu\nu}$ satisfies $g^{\mu\nu} T_{\mu\nu} = \partial^\mu V_\mu$ then

$$\oint_{\Sigma_{d-1}} * [\underbrace{(x^\mu T_{\mu\rho} - V_\rho)\, dx^\rho}_{\omega_D}] = D \cdot \quad :$$

$d^\dagger \omega_D \doteq * d * \omega_D = 0$ so generates symmetry (topological hypersurf. op.)

- If $g^{\mu\nu} T_{\mu\nu} = \partial^\mu \partial^\nu L_{\mu\nu}$ then ... $T \to \tilde{T}^\mu_\mu = 0$

$$\partial^\rho (x_\mu x^\nu \tilde{T}_{\nu\rho} - \tfrac{1}{2} x^2 \tilde{T}_{\mu\rho}) = 0 \implies \exists \text{ sym gen}$$

$$K_\mu = \oint_{\Sigma_{d-1}} * (\quad\qquad dx^\rho)$$

- $\{P_\mu, M_{\mu\nu}, D, K_\mu\} \simeq so(d-2,2)$
  Conformal algebra.

$$\begin{cases} D \circ K_\mu = -K_\mu \\ K_\mu \circ K_\nu = 0 \\ M_{\mu\nu} \circ K_\rho = (\mathcal{M}^{\text{vect}}_{\mu\nu})_\rho^{\ \sigma} K_\sigma \\ P_\mu \circ K_\nu = -2 g^{\mu\nu} D + 2 M_{\mu\nu} \end{cases}$$

— All unitary RG f.p. $\overset{?}{=}$ CFTs ?

— Local (unitary) CFTs (+ some 'mild' xtra assumps) have
- operator-state correspondence $\mathcal{O}_i \leftrightarrow |i\rangle$
- denumerable o-n basis of local primary ops — some $\mathcal{H}$

$$M_{\mu\nu} \mathcal{O}_i = (R^{\mathcal{O}}_{\mu\nu})_i^{\ j} \mathcal{O}_j \quad \text{irred.}$$
$$D\, \mathcal{O}_i = \Delta_{\mathcal{O}} \mathcal{O}_i$$
$$K_\mu \mathcal{O}_i = 0$$

diagonalization pos.

$$\langle i | j \rangle \sim \langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{g_{ij}\, \delta_{\Delta_i, \Delta_j}}{|x-y|^{2\Delta_i}}$$

- operator algebra (OPE)

fixed by conf. inv.

$$\mathcal{O}_i(x)\, \mathcal{O}_j(y) \sim \sum_k c_{ij}^{\ \ k}\, \mathcal{D}^k_{ij}(x-y, \partial_y)\, \mathcal{O}_k(y)$$

finite rad. convergence in correlators determined by nearest insertion assume discrete ...

$\{ R^{\mathcal{O}}, \Delta_{\mathcal{O}} \}$ = "spectrum of CFT"

$\{ c_{ij}{}^{k} \}$ = "structure consts"

$\Big\}$ "CFT data"

- Unitarity (pos. $\mathcal{H}$) $\Rightarrow$ inequalities on spectrum, reality conds C's
- Assoc. of op. algebra = "crossing relations"
- CFT data $\Rightarrow$ reconstruct $\forall$ CFT local correlators.

(· VOAs will have similar structure ...)

Recommend: David Simmons-Duffin CFT notes.
Recommend: Yu Nakayama arXiv:1302.0884

D. Supersymmetry

- WHY SUSY? SCFTs? : • Maximally symm QFTs
  • so good starting point to understand QFT analytically

~ QFTs can have add'tnl extension of Poincaré by adding fermionic generators Q^i $i = 1 \ldots \mathcal{N}$

$\Rightarrow$ Lie superalgebra

- spin-statistics $Q^i \in$ spinor Lorentz reps
- Haag-Sohnius-Lopuszanski thm: non-trivial interacts $\Rightarrow$ $Q^i \cdot Q^j \supset P^\mu$ (no $M^{\mu\nu}$), $Q^i \cdot P^\mu = 0$
- Classify algebras by # Q^i's $\equiv \mathcal{N}$ (no. Lor. irreps) su-transf.
- $\exists$ max $\mathcal{N}$ beyond which if $\exists$ massless particles then $\exists$ graviton:

$\#|Q^i_\alpha| \equiv Q \leq 16$:

$$\begin{pmatrix} d=3 & d=4 & d=5 & d=6 & \ldots & d=10 \\ \mathcal{N} \leq 8 & \mathcal{N} \leq 4 & \mathcal{N} \leq 2 & \mathcal{N} \leq (2,0) & & \mathcal{N} \leq (1,0) \\ & & & (1,1) & & \end{pmatrix}$$

— Super CFTs = SCFTs $\Rightarrow$ form simple Lie superalgebras:

$$\mathrm{Conf} + \left\{ \begin{matrix} Q \\ S \end{matrix} \right\} + R$$

Q: $\Delta = 1/2$; S: $\Delta = -1/2$

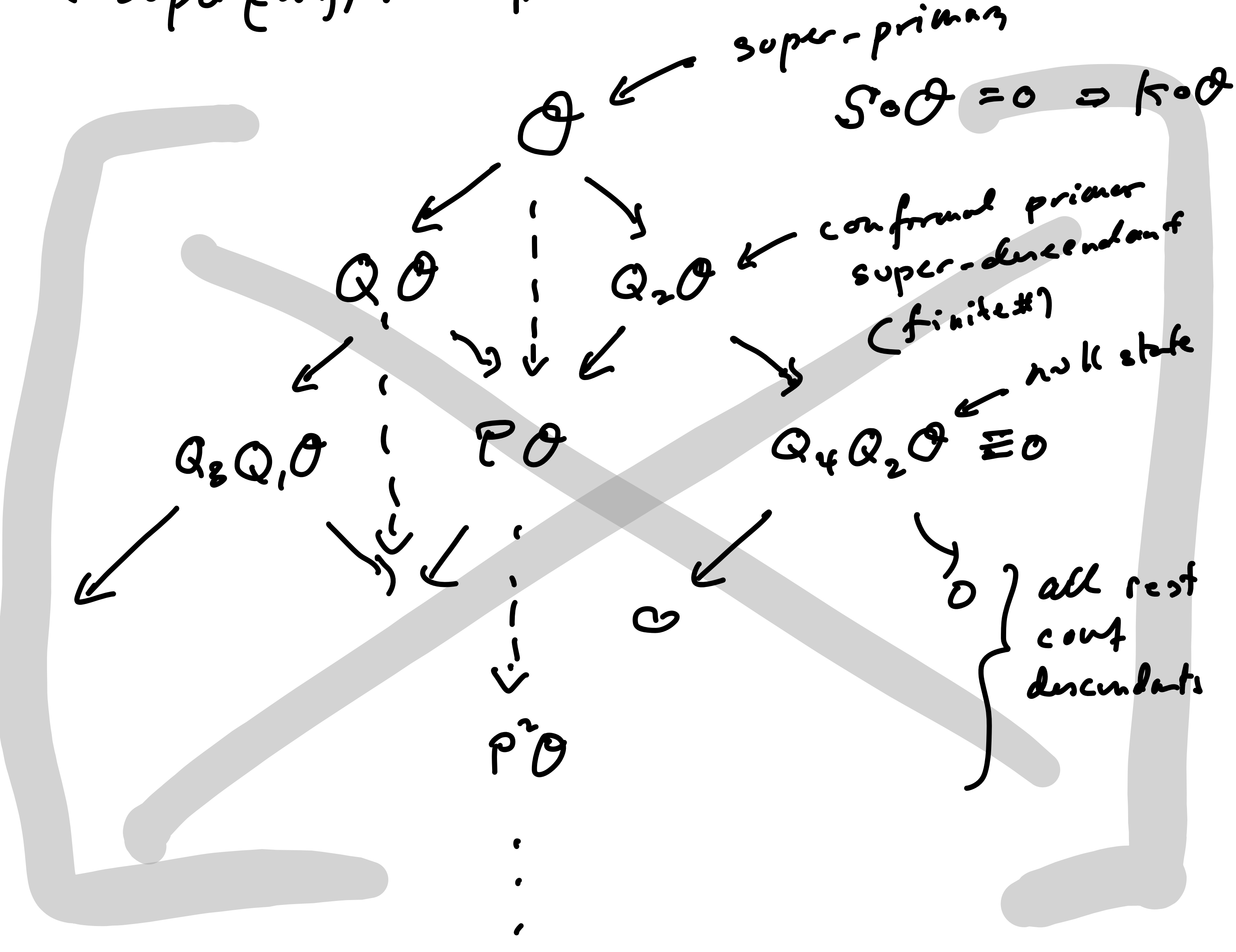
R: scalar, $\Delta = 0$ "R symm"

S: su-conf gens, $\#S = \#Q$

superLie (R | conf)

	$d=3$	$d=4$	$d=5$	$d=6$
	$osp(N\|4)$	$(p)su(2,2\|N)$	$F(4)$	$osp(8^*\|2N)$
	$N\leq 8$	$N\leq 4$		$N \leq 2$
$\#\lvert Q^i_\alpha\rvert =$	$2N$	$4N$	8	$8N$

- Super(conf) multiplets



- organize finite # conformal primaries in one super-plet
- But implications of SUSY/SCA go far beyond...

2. Moduli & parameter spaces, symmetries, dualities

A. Moduli spaces (vacua)

B. Parameter spaces (couplings)

C. Symmetries

D. Dualities

A. Moduli spaces (CFTs)

- Can have multiple vacua $|v\rangle$.

$\mathcal{M} \doteq \{|v\rangle\}$ "moduli space"

$\exists$ local ops $\phi_i(x)$ $i=1\cdots N$ such that

$\phi: \mathcal{M} \to \mathbb{R}^N$ embedding

$|v\rangle \mapsto \{\langle\phi_i\rangle_v\}$ ← "order parameters" = coords on $\mathcal{M}$

$\mathcal{M}$ ~ locally manifold structure $\simeq \mathbb{R}^n$
+ singularities in co-dim n (assumption!)

$\mathcal{M}^* = \mathcal{M} \setminus \{\text{singularities}\}$ ~ disjoint union of manifolds

$\langle\phi_i\rangle_v = \langle v|\phi_i(x)|v\rangle$ $\qquad P_\mu|v\rangle = M_{\mu\nu}|v\rangle = 0$

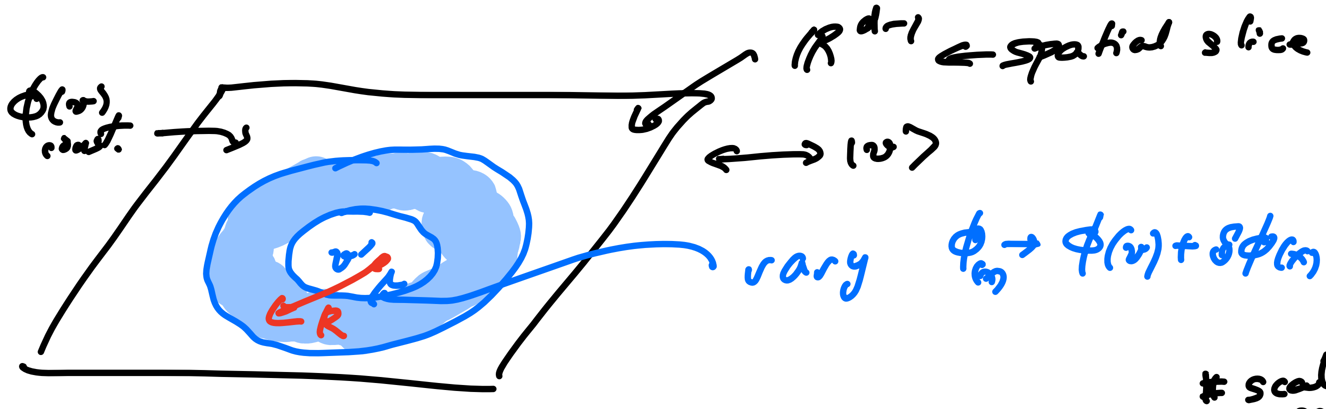
lect3

$\Rightarrow \partial_x\langle\phi_i\rangle_v = 0$ & $\langle\phi_i\rangle_v = 0$ unless $M_{\mu\nu}\phi_i = 0$

If $D|v\rangle = \Delta_v|v\rangle$ $\Delta_v > 0$ $\Rightarrow$ $|v_\lambda\rangle \doteq e^{\lambda D}|v\rangle$ also vacua $\forall \lambda\in\mathbb{R}$

scale inv. "spont. broken"

$\exists$ massless scalar free field "dilaton" (IR eff. act.)



Locality: energy $\propto \partial_\mu \delta\phi(x)$

scalar fields / moduli params

$i, j = 1 \cdots n$

- $\Rightarrow \mathcal{L} = g^{ij}(\phi(x)) \partial_\mu \phi_i(x) \partial^\mu \phi_j(x) + 4, 6, 8 \cdots$ derivatives

  $\{\phi_i\} : \mathbb{R}^d \to \mathcal{M}^*$ coords on $\mathcal{M}^*$

No 0-derivs ("potential") by assumption.

→ Describes n free, massless scalars

<u>Note</u>: energy cost to change $|v\rangle \to |v'\rangle$

$$\propto \lim_{R \to \infty} R^{d-2} = \infty \quad \text{if } d > 2$$

∴ For $d > 2$ each pt of $\mathcal{M}^*$

$\leftrightarrow$ distinct "superselection sector" ≠ CFT

($d = 2 \to \mathcal{M}$ "lifted by qu. fluct." to discrete set)

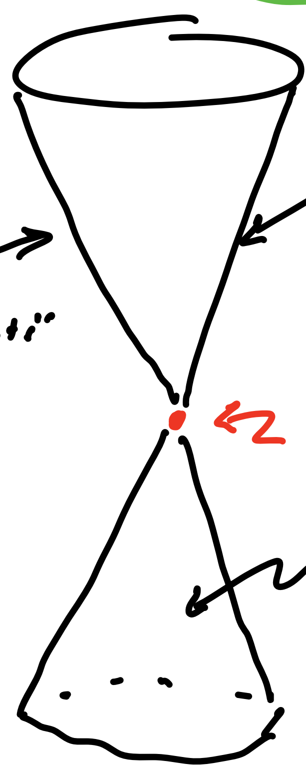
- $ds^2 \doteq g^{ij}(\phi)\, d\phi_i \cdot d\phi_j \Rightarrow$ Riemann. metric on $\mathcal{M}^*$ (unitarity $g > 0$)

distance ≃ energy cost.

∴ $\mathcal{M}$ = metric completion of smooth $\mathcal{M}^*$

Ex. 4d $\mathcal{N}=2$ SCFT "E_8 MN theory"

E_8 nil. orbit
E_8 isometry + $\mathbb{H}^\times$-action
"flavor" + dilatation + $SU(2)_R$



"Higgs br."
Eff. thry
massless "hypermultiplets"
$(\phi + \psi)$

$\simeq \mathbb{H}^{29}$ $\quad 29 = h^\vee - 1$
Hyperkahler cone

$\mathcal{O}_{min}$
E_8

scale-inv't

"Coul. branch"
massless "vectorplet"
$(\phi + \psi + A_\mu)$

$\simeq_{\mathbb{C}} \mathbb{C} \ni u$ $\quad$ Special Kahler
$ds^2 = |u|^{-5/3} du\, d\bar{u}$ $\quad$ $\mathbb{Z}_6$ orb.

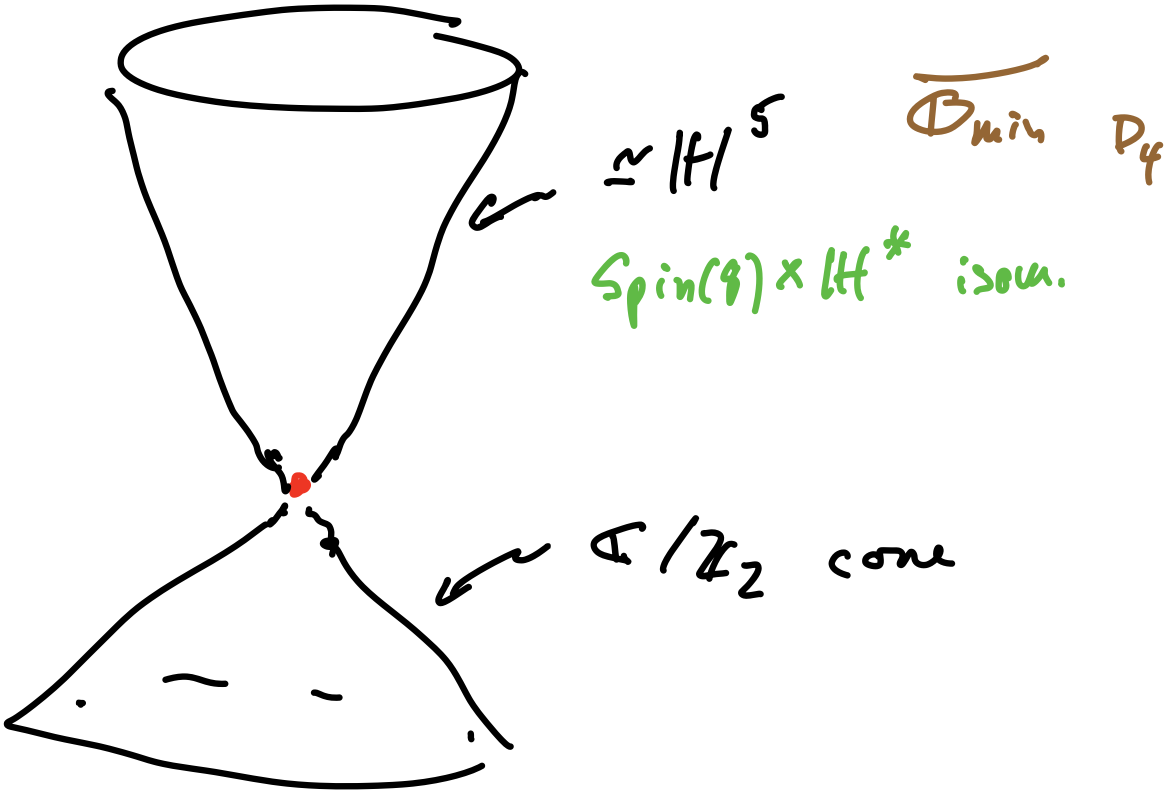
$\mathbb{C}^\times : u \mapsto \lambda u$
dilations + $U(1)_R$ act.

NB. Assumed that $Q|v\rangle = 0$
(Q = supertranslations) i.e. "susy not spont. broken"

Ex. 4d $\mathcal{N}=4$ $su(2)$ SYM $\quad \mathcal{M} = \mathbb{C}^3/\mathbb{Z}_2$

$$\simeq (\mathbb{H}^2 \times \mathbb{C}^2)/\mathbb{Z}_2$$

Ex. 4d $\mathcal{N}=2$ $su(2)$ $N_f=4$ QCD



B. Parameter spaces CFT = "Conformal mnfld"

- parameters = set of continuously varying numbers defining a QFT
  ≈ "coords on space of QFTs"

- "Conformal manifold" parametrizes continuously connected set of CFTs "$\mathcal{C}$"
  "(exactly) marginal deformations"
  "(exactly) dimensionless couplings"

- Conform. mnfld also metric $\langle O_i O_j \rangle \sim c_{ij}$
  $\Delta = d$ scalar primaries

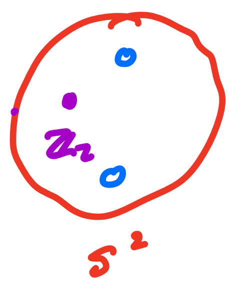
[NB exist ∞'ly many other parameters:

"relevant" (finitely many) ~ (energy) dim > 0, e.g. masses, potential terms

"irrelevant" (∞'ly many) ~ " " < 0, e.g. higher-deriv interactions]

- NB In $d=2$ CFTs "conf. mfld" = "moduli space"

Example: 4d $\mathcal{N}=4$ $SU(2)$ SYM

$\mathcal{C} =$ 

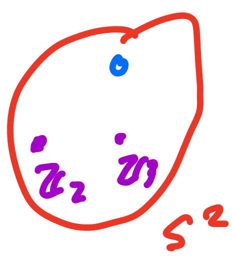
o = punctures @ ∞ ~ weak cplg

• = orbifold points

S-duality

$\Gamma_0(2)$

Example: 4d $\mathcal{N}=2$ $SU(2)$ $N_f=4$ QCD

$\mathcal{C} =$ 

$PSL(2,\mathbb{Z})$

Example: 4d $\mathcal{N}=2$ E_8 MN

$\mathcal{C} = \bullet$

trivial

C. Symmetries act unitarily on $\mathcal{H}$ of a given QFT.

[But not just any $M \in \mathcal{U}(\mathcal{H})$
need some s-t locality/localization properties]

$\Rightarrow$ So symms do <u>not</u> act on parameters!

But params may transform under symmetries of diff. theory w/ $\tau = \tau_*$: $\mathcal{C} \sim V/G$ organize on orbifold

$\Rightarrow$ Symm.s can/do act on moduli space, act as isometries (homothety for D) of $\mathcal{M}$ for "spont. broken" symms.

<u>Ex</u>: 4d $N=4$ su(2) sYM

symm $=$ $N=4$ sca $\simeq$ $psu(2,2|4)$

$\supset su(2,2|2) \oplus su(2)_f$

<u>Ex</u>: 4d $N=2$ su(2) $N_f = 4$ QCD

symm $=$ $(N=2 \text{ sca}) \oplus so(8)_f$

<u>Ex</u>: 4d $N=2$ E_8 MN

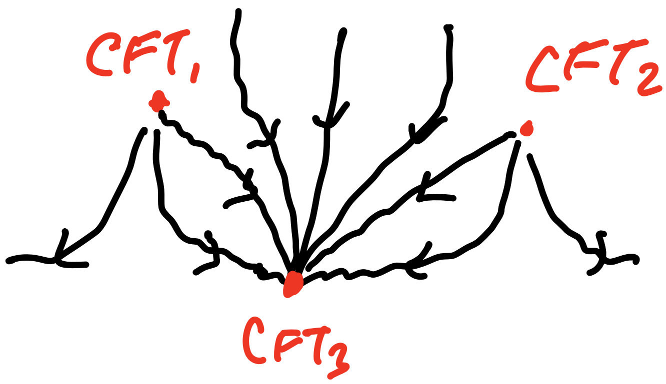
symm $= (N=2 \text{ sca}) \oplus E_8$

D. Dualities

2 kinds :

- "IR dualities" ~ "RG universality classes"

Ex ("Seiberg dualities" in 4d $\mathcal{N}=1$ sQCD)



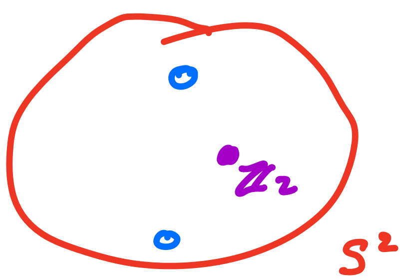
2 distinct CFTs flow to same CFT

- "S-duality"/"EM-duality"

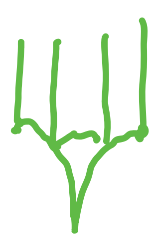
"two different CFTs are the same CFT"

Ex: • $\mathcal{N}=4$ $SU(2)$ sYM cplg $\tau \leftrightarrow$
• " $SO(3)$ sYM cplg $\tau' = -\frac{1}{\tau}$
(more $\tau \to \tau+1$, $\tau' \to \tau'+2$)

Right way: topology of conf. mfld $\mathcal{C}$

$\mathcal{C} \sim$ 

$\mathcal{N}=4$ $su(2)$ sYM example



$\pi_1^{orb}(\mathcal{C}) \cong \langle A, B \mid A^2 = 1 \rangle$

$\cong \Gamma_0(2) < PSL(2,\mathbb{Z})$

"S-duality group" of $SU(2)$ $N=4$ SYM.

~~3. CB's of 4d $N=2$ SCFTs~~ SKIP

4. Vertex alg(s) of 4d $N=2$ SCFTs

A. Twisting SCAs

B. VOA from "twisted Schur" twist

C. Generalized topological descent

D. VA of extended operators

4d $N=2$ SCFT has operator algebra

$$\mathcal{O}_i(x)\,\mathcal{O}_j(y) \sim \sum_k c_{ijk}\, \mathcal{D}_{ijk}(x-y, \partial_y)\, \mathcal{O}_k(z)$$

$x, y \in \mathbb{R}^{3,1}$

B. Find subalgebra $\{\mathcal{J}_a(z)\} \subset \{\mathcal{O}_i(x)\}$ s.t.

$$\mathcal{J}_a(z)\,\mathcal{J}_b(w) \sim \sum_c c_{abc}\, \mathcal{D}_{abc}(z-w, \partial_w)\, \mathcal{J}_c(w)$$

$z, w \in \mathbb{C} \subset \mathbb{R}^{3,1}$

which is a VOA! [Beem, .. 1312.5344]

D. Construct subalgebra of extended* op.s in 4d SCFTs

$$\{\mathcal{E}^{(n)}_a(z)\} \quad \text{s.t.} \quad \mathcal{E}^{(0)}_a(z) = \mathcal{J}_a(z) \quad \&$$

$$\mathcal{E}^{(n)}_a(z)\,\mathcal{E}^{(m)}_b(w) \sim \sum_c C^{nml}_{abc}\, \mathcal{D}^{nml}_{abc}(z-w,\partial_w)\,\mathcal{E}^{(l)}_c(w)$$

is a VA. [DCA, Lotito, Weaver 2211.04410]

"Extended ops" ≐ {Line op.s, Surface ops, Domain wall ops, ...}

Lect 4

A. "Twisting" SCFTs [Generalize Witten '88 "top. twist"]

$$SCA \doteq \mathfrak{b} \oplus \mathfrak{f}$$

$\mathfrak{b}$: bosonic Lie alg; $\mathfrak{f}$: fermionic, repn of $\mathfrak{b}$ ("supercharges")

B = Lie group of $\mathfrak{b}$

- $\Pi \in \mathfrak{f}$ s.t. $\Pi \circ \Pi = 0$ nilpotent
- Consider Π-cohomol. of: $\begin{cases} \text{loc. op.s} \in \text{SCFT} \\ \text{SCA itself} \end{cases}$

$$[\mathcal{O}(x)] \in H_\Pi \iff \begin{cases} \mathcal{O}(x) \in \{\text{loc. op. of SCFT}\} \\ \Pi\circ\mathcal{O}(x) = 0 \\ \mathcal{O}(x) \sim \mathcal{O}(x) + \Pi\circ\mathcal{O}(x) \end{cases}$$

("H^0_f" in BRST/DS lang. in Anne's talk.)

$$\mathfrak{z} = \{\ M \in \mathfrak{b} \ \ \& \ \ \Pi \circ M = 0 \ \}$$
$$\cup$$
$$\mathfrak{t} = \{\ M \in \mathfrak{b} \ \ \& \ \ M = \Pi \circ \mu, \ \ \mu \in \mathfrak{f} \ \}$$

$\Rightarrow$ $\mathfrak{z}$ & $\mathfrak{t}$ Lie (sub)algebras of SCA_0

$\Rightarrow$ $\Pi \circ (M \circ \zeta) = 0$ if $M \in \mathfrak{z}$

$\Rightarrow$ $[M \circ \zeta] = 0$ if $M \in \mathfrak{t}$

super-Jacobi idents

- $T \subset Z \subset B$ Lie groups of $\mathfrak{t} \subset \mathfrak{z} \subset \mathfrak{b}$

- Consider

Superconf. vacuum! $\mathfrak{b}|0\rangle = 0.$ (assume unique?)

$$\langle \zeta_1(x_1) \cdots \zeta_n(x_n) \rangle = \langle 0 | T[\zeta_1(x_1) \cdots \zeta_n(x_n)] | 0 \rangle$$

$\Rightarrow$ Functionals alg of classes $[\zeta_i] \in H_\Pi$

$$\langle \zeta_1 \cdots \zeta_{n-1} (\Pi \circ \theta_n) \rangle = \langle 0 | \cancelto{0}{\Pi}\, \zeta_1 \cdots \zeta_{n-1} \theta_n | 0 \rangle + \sum_{j=1}^{n} \langle 0 | \zeta_1 \cdots \cancelto{0}{(\Pi \circ \zeta_j)} \cdots \zeta_{n-1} \theta_n | 0 \rangle + \langle 0 | \zeta_1 \cdots \zeta_{n-1} \theta_n \cancelto{0}{\Pi | 0 \rangle}$$

$\Rightarrow$ $\langle \zeta_1 \cdots \zeta_{n-1} (M \circ \zeta_n) \rangle = 0$ if $M \in \mathfrak{t}$

- If $g \in B$

$$g \circ \mathcal{S}(x) = \mathcal{S}^{g}(g \circ x)$$

field transf w/in Lorentz ⊗ R-symm rep

conf grp actn

- If $g \in Z$, $\mathcal{S}(x) \in H_{\Pi} \Rightarrow g \circ \mathcal{S}(x) \in H_{\Pi}$

- Configuration space $\mathcal{C} \subset \mathbb{R}^d$ ← s-t.

$$\mathcal{C} \doteq B \circ 0 \quad \text{(orbit of } B\text{-act on } \mathbb{R}^d)$$

($0 \in \mathbb{R}^d$)

$\Rightarrow \langle \mathcal{S}_1(x_1) \cdots \mathcal{S}_n(x_n) \rangle \doteq G_n : H_{\Pi}^n \times (\mathcal{C}^n)^* \longrightarrow \mathbb{C}$

& $\Rightarrow \langle \mathcal{S}_1(x_1) \cdots g \circ \mathcal{S}_n(x_n) \rangle = \langle \mathcal{S}_1(x_1) \cdots \mathcal{S}_n(x_n) \rangle$

if $g \in T$

$\Rightarrow$ G_n only depend on $\mathcal{C}/T$,

$\{G_n\}$ define a sub-QFT = "Π-twisted FT"

<u>Ex</u> If Π s.t. $P_\mu \in \mathfrak{t} \quad \forall \mu$

$$\mathcal{C} = \mathbb{R}^d \qquad \mathcal{C}/T = \text{point}$$

$G_n(x_1 \cdots x_n)$ indep't of $x_i \Rightarrow$ topological QFT e.g. "Donaldson - Witten twist" in 4d $\mathcal{N}=2$ SCA

B. VOA of Beem et al

Ex If $\begin{cases} \Pi_+ \doteq Q^1_\alpha + \tilde{S}^{2\dot\alpha} \\ \text{or } \Pi_- \doteq \tilde{Q}^1_{\dot\alpha} + S^{2\alpha} \end{cases}$ "twisted Schur" in 4d $\mathcal{N}=2$ SCA

$su(2,2|2)$

$$\underline{t_\pm} = \widehat{sl_2} \oplus r\mathbb{R} \oplus m_\pm$$

$$\mathfrak{z}_\pm = sl_2 \oplus t_\pm$$

$= \langle L_{-1}, L_0, L_1 \rangle$

Translations:

$$sl_2 \supset P_z \doteq P_1 + iP_2 \doteq L_{-1} = \partial$$

$$\widehat{sl_2} \supset P_{\bar z} \doteq P_1 - iP_2$$

$$m_+ \supset P_+ \doteq P_3 + P_4$$ time-like

$$m_- \supset P_- \doteq P_3 - P_4$$

$$\mathcal{C}_\pm = \underbrace{\mathbb{C}}_{z\bar z \text{ plane}} \times \underbrace{\mathbb{R}_\pm}_{\perp \text{ light-like direction}} \qquad \mathcal{C}_\pm / T_\pm \simeq \underbrace{\mathbb{C}_z}_{z \text{ line}}$$

$$G^n_\pm : (\mathbb{C}^n_z)^* \longrightarrow \mathbb{C}$$ holomorphic in z_i (possible poles @ $z_i = z_j$)

$\Rightarrow$ Vertex algebra

- Unitarity of 4d $\mathcal{N}=2$ SCFT $\Rightarrow$

(1) $H_{\pi_+} \cong H_{\pi_-} \Rightarrow G_+^h = G_-^h$ same VA

✓ (3) L_0-grading $\Delta \in \frac{1}{2}\mathbb{Z}_{\geq 0}$

✓ (4) $\exists\ T(z) \in VA$

w/ $T(z)T(0) \sim \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}$

? (5) $T(z) \doteq \sum_n L_{-n} z^{n-2}$

$\Rightarrow \langle L_{-1}, L_0, L_1 \rangle = sl_2 \subset Z$

? (6) $\dim V_\Delta < \infty \quad \forall \Delta$

VOA

C. Descent [Generalizes Witten '88 "top. descent"]

- Procedure to construct new H_π cohomology classes from $[\mathcal{S}(x)]$.

$$D^M[\mathcal{S}(x)] \doteq \int dx\, \mu \circ e^{\alpha M} \circ \mathcal{S}(x)$$

when $\mathfrak{t} \ni M = \pi \circ \mu \quad (\mu \in \mathfrak{f})$

$\mathfrak{t}$ π-exact bosonic generators

$$\int d\alpha = \int_{-\infty}^{\infty} d\alpha \quad \text{if non-compact gen } M$$

$$= \int_{S^1} d\alpha \quad \text{" compact " } M$$

- If $D^M[\mathcal{J}(x)]$ converges (not always: need $M \circ x \neq 0$)

- $\Rightarrow \Pi \circ (D^M[\mathcal{J}(x)]) = 0$
  (up to boundary terms if M non-compact!)

so $[D^M[\mathcal{J}(x)]] \in H_\Pi$. (i.e., enlarge defn H_Π)

- $\Rightarrow \ldots \mathcal{C}(D^M[\mathcal{J}(x)]\text{'s}) \simeq \mathcal{C}(\mathcal{J}(x)\text{'s})$

config. space

"descent"

So adding $\mathcal{D}(M,x) \doteq D^M[\mathcal{J}(x)]$'s
extends set of $\langle \mathcal{J}_1(x_1) \cdots \mathcal{J}_n(x_n) \rangle$ correls
to $\langle \mathcal{J}_1(x_1) \cdots \mathcal{J}_n(x_n)\, \mathcal{D}_1(M_1,x_1) \cdots \mathcal{D}_m(M_m,x_m) \rangle$
$\Rightarrow$ defines a new kind of Π-twisted FT

extended Π-twisted FT $\supset$ local Π-twisted FT

• Can iterate descent procedure:

$$\mathcal{D}^{(n)}(x) \simeq D^{M_1} D^{M_2} \cdots D^{M_n}[\mathcal{S}(x)] \approx \text{n-dim'l extended op. in } \Pi\text{-cohom.}$$

$(\mathcal{D}^{(0)} \doteq \mathcal{S})$

Ex Π = Donaldson–Witten top twist of 4d $\mathcal{N}=2$ QFT

$D^{P_\mu}[\mathcal{J}(x)]$ ~ top line (fermionic)

$D^{P_\mu} D^{P_\nu}[\quad]$ ~ " surface

.. ~ domain walls

& extended Π-twist TFT computes Donaldson invariants...

D. VA of extended ops in 4d $\mathcal{N}=2$ SCFTs

• Recall $\mathfrak{t}_{\Pi_\pm} = \widehat{\mathfrak{sl}_2} \oplus \mathfrak{u}(1)_R \oplus \mathfrak{m}_\pm$

$$\mathfrak{m}_\pm = \langle P_\pm, M_{\pm z}, M_{\pm \bar{z}}, K_\pm \rangle$$

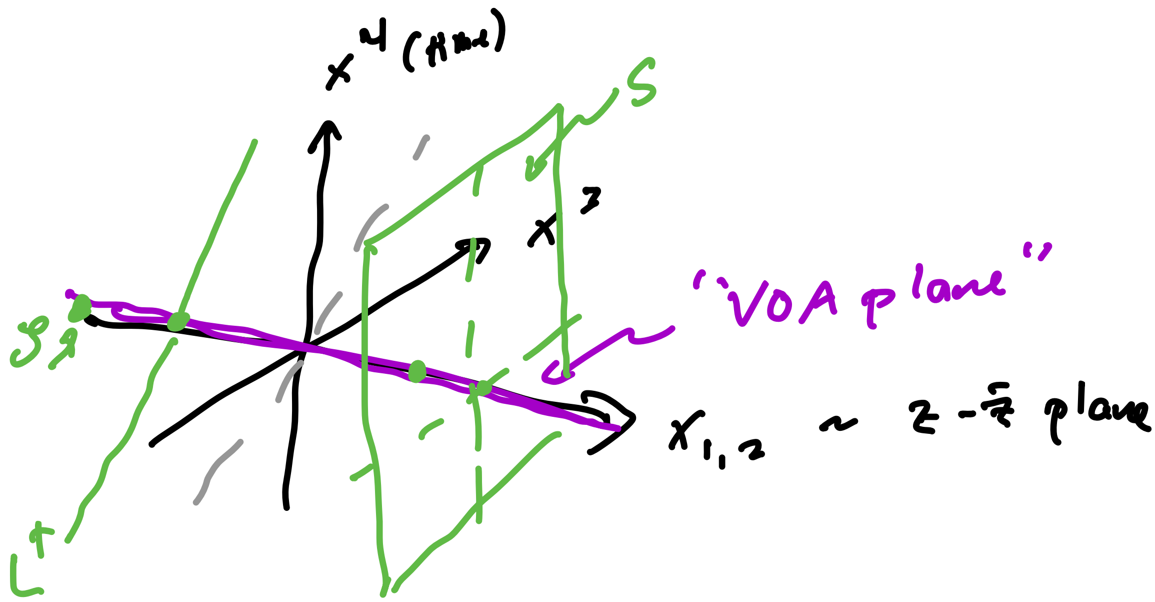
* Key fact: $D_+^M[\mathcal{D}]$ also in Π_- cohom! (& $+ \leftrightarrow -$)

$\mathcal{J} \in H_{\Pi_+} \Rightarrow \in H_{\Pi_-}$ b/c unitarity (pos. def. inner prod on $\mathcal{H}$)

No such for $\mathcal{D}$ → "algebraic accident" ??

$$
\begin{tikzcd}
 & & \mathfrak{g} \arrow[dl, "D_+^{p_+}"'] \arrow[dr, "D_-^{p_-}"] & & \\
 & L^+_{\mathfrak{g}} \arrow[dl, "D_-^{M-3}"'] \arrow[dr, "D_-^{p_-}"] & & L^-_{\mathfrak{g}} \arrow[dl, "D_+^{p_+}"'] \arrow[dr, "D_+^{M+3}"] & \\
\hat{S}_R & & S_{\mathfrak{g}} & & \hat{S}_L
\end{tikzcd}
$$

$\hat{W}$'s ...



$* \quad sl_2 \sim \langle L_{-1}, L_0, L_1 \rangle \circ D[\mathfrak{g}] = sl_2 \circ \mathfrak{g}$

i.e. $L_{-1} = \frac{\partial}{\partial z}$

$L_0 = h$ same

$L_{+1} = 0$ on primary $D[\mathcal{S}]$

* Then $\{ \mathcal{D}^{(n)} \}$ generate (local) VA

- does not close on $\mathcal{D}^{(n)}$'s:

$$\mathcal{D}_i(z)\, \mathcal{D}_j(w) \sim \sum_k C_{ijk}\, \mathcal{E}_k(w)$$

new (non-descent) fields

- in 4d "$\mathcal{E} \sim :\mathcal{D}\mathcal{D}':$" etc

Example 4d $\mathcal{N}=2$ SCFT = "free hypermultiplet"

• VOA:

$$\begin{cases} q_I(z)\, q_J(w) \sim \dfrac{-\epsilon_{IJ}}{z-w} & I,J=1,2 \\ T(z)\, q_J(w) \sim \left(\dfrac{1/2}{(z-w)^2} + \dfrac{\partial_w}{(z-w)} \right) q_J(w) \\ T(z)\, T(w) \sim \left(\dfrac{-1/2}{(z-w)^4} + \dfrac{2}{(z-w)^2} + \dfrac{\partial_w}{(z-w)} \right) T(w) \end{cases}$$

Strongly generated by $\{ q_I \}$.

• extended VA :

$$X_I(z)\, Y_J(w) \sim \epsilon_{IJ} \frac{a}{z} \qquad X, Y \in \{q, S_q, L^\pm_q\}$$

a

$X \backslash Y$	q	S_q	L^+_q	L^-_q
q	-1	-1		
S_q	-1	-1		
L^+_q				-1
L^-_q			1	

!?

degenerate

$$X_T(z)\, Y(w) \sim \left[\frac{1/2}{(z-w)^2} + \frac{\partial_w}{(z-w)}\right] V(w)$$

$X_T \in \{T, S_T, L^\pm_T\}$ $\qquad$ $Y, V \in \{q, S_q, L^+_q, L^-_q\}$

V

$X_T \backslash Y$	q	S_q	L^+_q	L^-_q
T	q	q	0	0
S_T	S_q	S_q	L^+_q	L^-_q
L^+_T	L^+_q	L^+_q		q
L^-_T	L^-_q	L^-_q	q	

not conf. vec...

$$X_T(z)\, Y_T(0) \sim \sim \frac{c/2}{z^4} + \frac{2(uv)}{z^3} + \frac{\partial(uv)}{z^2} + \frac{\partial^2(uv)}{z}$$

$X_T, Y_T \in \{T, S_T, L^\pm_T\}$ $\qquad + \frac{2(u\wedge v')}{z^2} + \frac{\partial(u\wedge v')}{z} + \frac{(u'v')}{z}$

$c \in \{\pm 1, 0\}$

$u, v \in \{q, S_q, L^\pm_q\}$ w/

$$(uv) \doteq -\frac{1}{4}\epsilon^{IJ} :U_I V_J:$$

$$(u\wedge v') \doteq +\frac{1}{4}\epsilon^{IJ} :(U_I \partial_z V_J - \partial_z U_I V_J):$$

$$(u'v') \doteq +\epsilon^{IJ} :\partial_z U_I \partial_z V_J:$$

$$(uv) = -(-)^{|u|\cdot|v|}(vu) \qquad |u| = \begin{cases} 0 & \text{boson} \\ 1 & \text{fermion} \end{cases}$$

$$(u\wedge v') = +(-)^{|u|\cdot|v|}(v\wedge u')$$

$$(u'v') = -(-)^{|u|\cdot|v|}(v'u')$$

Note $\qquad (c\ u\ v)$

$X_T \backslash Y_T$	T	$S[T]$	$L^+[T]$	$L^-[T]$
T	$(1,q,q)$	$(1,q,S[q])$	$(0,q,L^+)$	$(0,q,L^-)$
$S[T]$	$(1,S,q)$	$(1,S,S)$	$(0,S,L^+)$	$(0,S,L^-)$
$L^+[T]$	$(0,L^+,q)$	$(0,L^+,S)$	$(0,L^+,L^+)$	$(1,L^+,L^-)$
$L^-[T]$	$(0,L^-,q)$	$(0,L^-,S)$	$(-1,L^-,L^+)$	$(0,L^-,L^-)$